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Tomographic reconstruction using Cesaro-means and Newman-Shapiro operators

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Abstract

Tomography is well known because of its many applications. Although theoretically solved, the numerical implementation of tomographic reconstruction algorithms is still a difficult problem. In this article the numerical implementation of a reconstruction method using Cesaro-means and Newman-Shapiro operators is described. The key point herein is the use of suitable quadrature formulae on the sphere. It turns out that in the context described product Gaussian formulae are best suited. The algorithm is tested at the so called Shepp-Logan phantom which is a three dimensional model of a human head.

1 Introduction and notation

The problem in tomography is to reconstruct a function F from its Radon transform sufficiently well. Since certain classes of functions can be expanded into series of orthogonal polynomials it is essential to exploit the action of the Radon transform on orthogonal polynomials and on polynomials in general.

This approach is the more interesting since the inverse of the Radon transform for polynomials is known explicitly.

The convergence of orthogonal expansions to the given function is often achieved only by applying a summability method. The application of such methods can be interpreted as a kind of "filter technique" which is necessary for sufficiently good reconstructions. The combination of an expansion of the function and the application of suitable summability methods leads to promising reconstruction algorithms.

In this article two examples for a summability method and their implementation are presented — the Cesaro-means and Newman-Shapiro-means. After some introductory remarks on Laplace-series at the end of this section, in Section 2 the theory of summability methods needed here is presented. In Section 3 this theory is applied to the reconstruction of functions from their Radon transform. Section 4 describes the numerical implementation of the reconstruction formula which is tested on the so called Shepp–Logan phantom of a head in Section 5.

In this article the following notation is used. Let B^r denote the unit ball in $\mathbb{R}^r$, S^{r-1} denote the unit sphere and $Z^r := [-1,1] \times S^{r-1}$. xy denotes the Euclidean product of $x, y \in \mathbb{R}^r$.

The spaces of restrictions of r–variate polynomials, homogeneous polynomials and homogeneous harmonic polynomials of degree $\mu \in I\!N_0$ onto a subspace $X \subset I\!R^r$ ($X = S^{r-1}$ or $X = B^r$) are denoted by $I\!P_\mu^r(X)$, $\overset{*}{I\!P}{}_\mu^r(X)$, $\overset{*}{I\!H}{}_\mu^r(X)$, respectively. The space $C(S^{r-1})$ of all continuously differentiable functions is provided with the inner product $< F, G >:= \int_{S^{r-1}} F(x)G(x)dx$. The surface measure of the sphere is denoted by $\omega_{r-1} =< 1, 1 >$.

Let C_μ^λ denote the Gegenbauer polynomials of degree μ and index λ and $\tilde{C}_\mu^\lambda = C_\mu^\lambda / C_\mu^\lambda(1)$ the normalized Gegenbauer polynomials. The reproducing kernel function of $\overset{*}{I\!H}{}_\mu^r(S^{r-1})$ is given by $G_\mu(xy) = \dfrac{2\mu + r - 2}{(r-2)\omega_{r-1}} \cdot C_\mu^{\frac{r-2}{2}}(xy)$, the normalized reproducing kernel $\tilde{G}_\mu$ is defined by $\tilde{G}_\mu := G_\mu / G_\mu(1)$.

Let $Y \in \{C(S^{r-1}), L^2(S^{r-1}), L^p(S^{r-1})\}$. For $f \in Y$ let

$$L(f,x) = \sum_{\nu=0}^{\infty} \big(\Lambda_\nu f\big)(x) = \sum_{\nu=0}^{\infty} \int_{S^{r-1}} f(y) G_\nu(xy) dy \tag{1.1}$$

be the Laplace–series of f, where $\big(\Lambda_\nu f\big)(x) := \int_{S^{r-1}} f(y) G_\nu(x,y) dy$ is the orthogonal projection of f onto $\overset{*}{I\!H}{}_\nu^r(S^{r-1})$ and the partial sums $L_\mu(f,x) = \sum_{\nu=0}^{\mu} \big(\Lambda_\nu f\big)(x)$ are the orthogonal projections of f onto $I\!P_\mu^r(S^{r-1})$.

Whereas for $Y = L^2(S^{r-1})$ it is known that the partial sums $L_\mu(f,x)$ converge to f in norm, no convergence is obtained for $Y = C(S^{r-1})$ or $Y = L^p(S^{r-1})$ for $p \geq 2 + \dfrac{r}{r-2}$ and $p \leq 2 - \dfrac{2}{r}$ (see e.g. [1]p.211). Applying a summability method the situation changes.

2 Summability methods

Let $A = (a_{\mu\nu})_{\mu,\nu \in I\!N_0}$ be an infinite matrix for which the elements $a_{\mu\nu} \in I\!R$ fulfil the following properties.

(i) $a_{\mu\nu} = 0$ for $\nu > \mu$,
(ii) $\lim_{\mu\to\infty} a_{\mu\nu} = 1$ for $\nu \in \{0, 1\}$,
(iii) $K_\mu(\xi) \geq 0$ for $-1 \leq \xi \leq 1$, where $K_\mu := \sum_{\nu=0}^{\mu} a_{\mu\nu} G_\nu$.

If with the aid of a summability method the kernel G_ν in (1.1) is substituted by a kernel

$$K_\mu = \sum_{\nu=0}^{\mu} a_{\mu\nu} G_\nu \tag{2.1}$$

then the operator L^A defined by the transformed series

$$L^A(f,x) = \lim_{\mu\to\infty} \int_{S^{r-1}} f(y) K_\mu(x,y) dy \tag{2.2}$$

can be shown to converge pointwise to the identity provided that for the kernel K_μ the properties (i)–(iii) of the matrix A are valid.

Remark 2.1 *The coefficients $a_{\mu\nu}$ can be obtained from*

$$a_{\mu\nu} = \big(L_\mu^A \tilde{G}_\nu(t.)\big)(t) = \int_{S^{r-1}} \tilde{G}_\nu(tx) K_\mu(tx) d\omega(x), \quad t \in S^{r-1}.$$

For A being the matrix of the Cesaro-means the proof was given by Kogbetlianz [4] first. Berens et al. [1] give a proof for Cesaro-means as well as for Abel-Poisson-means. They also prove results on the order of convergence and the corresponding saturation classes. The convergence proof for Newman-Shapiro operators ($Y = C(S^{r-1})$) can be found in Reimer [7].

2.1 Cesaro-means

For Cesaro–means the coefficients $a_{\mu\nu}$ in the summability method have to be chosen as

$$a_{\mu\nu} = \frac{(1)_\mu}{(k+1)_\mu} \frac{(k+1)_{\mu-\nu}}{(1)_{\mu-\nu}}, \tag{2.3}$$

where $(p)_q = p \cdot (p+1) \cdot \ldots \cdot (p+q-1)$ denotes the Pochammer symbol. Then the kernels K_μ in (iii) take on the form

$$K_\mu = \frac{(1)_\mu}{(k+1)_\mu} \sum_{\nu=0}^{\mu} \frac{(k+1)_{\mu-\nu}}{(1)_{\mu-\nu}} G_\nu. \tag{2.4}$$

Convergence of the transformed Laplace-series (2.2) is valid for $k > (r-2)/2$; for $k \geq r-1$ the operators even are positive (see Kogbetlianz [4]).

2.2 Newman-Shapiro summability method

In [8] Reimer considers kernel polynomials

$$K_{2\nu+1}(\xi) := K_{2\nu}(\xi) := g_{\nu+1} \left[\frac{G_{\nu+1}(\xi)}{\xi - \eta_{\nu+1}} \right]^2 \tag{2.5}$$

as used by Newman-Shapiro [5]. Here, $\eta_{\nu+1}$ is the largest root of $G_{\nu+1}$ and

$$g_{\nu+1} = (r-2)\omega_{r-1} \cdot \frac{1-\eta_{\nu+1}^2}{(2\nu+r)^2} \binom{\nu+r-2}{r-3}^{-1} = \frac{1-\eta_{\nu+1}^2}{2\nu+r} \cdot \frac{1}{G_{\nu+1}(1)}. \tag{2.6}$$

The coefficients $a_{\mu\nu}$ in the Newman-Shapiro operators can be calculated to be

$$\begin{aligned} a_{\mu\nu} \quad = \quad & g_{\nu+1} \cdot \sum_{j=0}^{\nu} \sum_{l=0}^{\nu} \frac{(2\nu+r)^2}{(\nu+1)^2} \cdot \frac{\tilde{G}_j(\eta_{\nu+1}) \tilde{G}_l(\eta_{\nu+1})}{\big(\tilde{G}_\nu(\eta_{\nu+1})\big)^2} \cdot \frac{(j+\lambda)(l+\lambda)}{\omega_{r-1}\lambda^2} \\ & \cdot \sum_{k=0}^{\min\{j,l\}} \frac{(\lambda)_k}{(1)_k} \frac{(\lambda)_{j-k}}{(1)_{j-k}} \frac{(\lambda)_{l-k}}{(1)_{l-k}} \frac{(1)_{j+l-2k}}{(2\lambda)_{j+l-2k}} \frac{(2\lambda)_{j+l-k}}{(\lambda+1)_{j+l-k}} \cdot \delta_{\nu,j+l-2k}, \end{aligned} \tag{2.7}$$

where $\delta_{\nu,j+l-2k}$ denotes the Kronecker delta and $\lambda = \frac{r-2}{2}$.

The matrix A defined by the Newman-Shapiro operators fulfils the properties (i)–(iii) (see Reimer [8]).

Remark 2.2 *The corresponding partial sum operators L_μ^A are nonnegative with positive $a_{\mu\nu}$. For continuous and differentiable functions even more is valid (see Reimer [8]): whereas for continuous functions the approximation error is of order $O(\mu^{-1})$, functions $F \in C^j(S^{r-1})$, $j \in \{1,2\}$, have an error of order $O(\mu^{-j})$.*

3 Application to tomography

The Radon transform $\mathcal{R} : C(B^r) \to C(Z^r)$ is defined by

$$(\mathcal{R}F)(s,t) := \int\limits_{\substack{v \perp t \\ v^2 \le 1-s^2}} F(st+v)dv, \quad F \in C(B^r), \quad (s,t) \in Z^r, \tag{3.1}$$

which means that the Radon transform $\mathcal{R}$ of F is determined by integrating F over all hyperplanes of dimension $r-1$. This map can also be defined for functions in $L^1(I\!R^r)$, $L^2(B^r)$, the Schwartz space $\mathcal{S}(I\!R^r)$ or some Sobolev spaces. $\mathcal{R}$ is continuous on all of these spaces, whereas the inverse $\mathcal{R}^{-1}$ is only continuous on $\mathcal{S}(I\!R^r)$ and on the Sobolev spaces.

For polynomials it is known that

$$\big(\mathcal{R}C_\mu^{\frac{r}{2}}(a.)\big)(s,t) = \tilde{C}_\mu^{\frac{r}{2}}(s)C_\mu^{\frac{r}{2}}(at), \quad a \in S^{r-1}, \quad (s,t) \in Z^r \tag{3.2}$$

(see Davison, Grünbaum [2]) and, more generally,

$$\big(\mathcal{R}P_m\big)(s,t) = \tilde{C}_\mu^{\frac{r}{2}}(s)P_m(t), \quad (s,t) \in Z^r, \tag{3.3}$$

where the polynomials $P_m \in \overset{*}{I\!P}{}_\mu^r (S^{r-1})$ are generated by the Gegenbauer polynomials, i.e. $\frac{1}{\omega_{r-1}}C_\mu^{\frac{r}{2}}(ax) = \sum_{|m|=\mu} a^m P_m(x)$. These polynomials P_m, $|m| = \mu$, are known to constitute a basis for $\overset{*}{I\!P}{}_\mu^r (S^{r-1})$.

Let $V_\mu^r := span\{P_m : |m| = \mu\}$. Since the Gegenbauer polynomials $C_\nu^{\frac{r}{2}}$ can also be interpreted as the reproducing kernel of $\overset{*}{I\!H}{}_\mu^{r+2}(S^{r+1})$, the orthogonal projection F_ν of $F \in C(B^r)$ onto $V_\nu^r(B^r)$ can be identified with the orthogonal projection of F onto $\overset{*}{I\!H}{}_\nu^{r+2}(S^{r+1})$ (see Reimer [7] for details). Thus the theory of Laplace series can be used here for the reconstruction of F from its Radon transform.

Let A be a matrix transformation as introduced in Section 2 and let F_ν be the orthogonal projection of F onto $V_\nu^r(B^r)$. Then according to the summability theory of Laplace series $F = \lim_{\mu\to\infty} \sum_{\nu=0}^{\mu} a_{\mu\nu} F_\nu$. Since the Radon tranform is linear and continuous there is $\mathcal{R}F = \lim_{\mu\to\infty} \sum_{\nu=0}^{\mu} a_{\mu\nu} \mathcal{R}F_\nu$.

It can be shown that (see Reimer [7])

$$F_\nu(x) = \lambda_{\nu,r} \frac{\omega_{r-2}}{r-1} \int_{Z^r} \big(\mathcal{R}F\big)(s,t) C_\nu^{\frac{r}{2}}(s) C_\nu^{\frac{r}{2}}(tx) d(s,t), \tag{3.4}$$

where

$$\lambda_{\nu,r} = \frac{(r-1)\, C_\nu^{\frac{r}{2}}(1)}{\omega_{r-1}\cdot\omega_{r-2}} \int_{-1}^{1} \Big(C_\nu^{\frac{r}{2}}(s)\Big)^2 (1-s^2)^{\frac{r-1}{2}} ds = \frac{2\nu + r}{\omega_{r-1}^2}. \tag{3.5}$$

¿From this, after some lengthy calculation using the adjoint operator of $\mathcal{R}$ (which essentially is the inverse operator of $\mathcal{R}$), the reconstruction formula follows

$$F(x) = \lim_{\mu\to\infty} \sum_{\nu=0}^{\mu} a_{\mu\nu}\lambda_{\nu,r} \int_{S^{r-1}} \int_{-1}^{1} (\mathcal{R}F)(s,t) C_\nu^{\frac{r}{2}}(s) C_\nu^{\frac{r}{2}}(tx) ds dt. \tag{3.6}$$

Because of the identification of the orthogonal projection of F onto $V_\nu^r(B^r)$ and onto $\overset{*}{\mathbb{H}}{}_\nu^{r+2}(S^{r+1})$, convergence of the Cesaro-means follows for $k > r/2$, and positivity of the operators is valid for $k \geq r+1$. For the same reason the coefficients $a_{\mu\nu}$ in the Newman-Shapiro summability method have to be calculated for $\lambda = \frac{(r+2)-2}{2} = \frac{r}{2}$.

4 Numerical implementation

For the reconstruction of F formula (3.6) was used. As soon as the Radon transform of F is known, the numerical implementation in principle reduces to a stable evaluation of the Gegenbauer polynomials and a suitable approximation of the integrals in (3.6). The Gegenbauer polynomials were evaluated by their recurrence relation (see Szegö [11]) which is known to be numerically very stable. The coefficients $a_{\mu\nu}$ for the Cesaro-means and the Newman-Shapiro operators were computed with the aid of formula (2.3) and (2.7), respectively. The factor $\lambda_{\nu,r}$ was obtained by (3.5). Since the calculation of $a_{\mu\nu}$ for the Newman-Shapiro operators is very time consuming (more than 10 hours for $\mu > 100$) these coefficients were stored before the main computation was started.

Since the integrand in (3.6) is a polynomial of degree $\nu + 2$ with respect to s (see (3.6) together with (5.1)), $\int_{-1}^{1} ..ds$ was approximated by a Gaussian-Legendre quadrature of degree $\mu/2 + 1$. This choice ensures that for the evaluation of $\mathcal{R}F(s,t)$ enough evaluations with respect to s are performed and that the integral is evaluated exactly within numerical precision.

For the quadrature on S^{r-1} first an interpolatory quadrature as introduced in [6]p.132 was used. The weights of such a quadrature formula are obtained as solutions of a linear system of equations $GA = e$, where $e = (1,\ldots,1)^T \in \mathbb{R}^N$, $N = \dim \overset{*}{\mathbb{P}}{}_\mu^r\ (S^{r-1})$, $A = (A_1,\ldots,A_N)^T$ the vector of weights and

$$G = \frac{1}{\omega_{r-1}} (C_\mu^{\frac{r}{2}}(x_j x_k) + C_{\mu-1}^{\frac{r}{2}}(x_j x_k))_{j,k=1}^{N}.$$

The points were chosen to be regularly distributed on latitudes of the sphere.

For $\mu \geq 70$ in the computation of the weights computational problems occured because of a lack of memory. Apart from this problem, several weights turned out to be negative which led to oscillations of the reconstruction. Therefore, this interpolatory quadrature was substituted by a product–Gauss formula for the sphere S^{r-1} as suggested by Stroud [10]p. 41. The points and weights of the Gaussian quadrature were computed by the MATLAB program `qrule.m` which is available via internet from the Mathworks Inc. The number of points of the product Gauss formula is $N = 2M^{r-1}$ where $M = \mu/2 + 1$ is the number of points used in each direction, i.e. $N = 2M^2$ for $r = 3$.

All codes for computation were written in MATLAB 6. The actual computation took place on a SUN Ultra10 with 256 MB main memory, 691 MB virtual memory and SUN OS operating system release 5.7. To increase the computatinal speed all parts of the MATLAB code were written with as few for-loops as possible. This gave an improvement in speed of a factor > 500.

5 Computational results

The theoretical results have been applied to the so called Shepp-Logan phantom which is usually used as a test function for tomographic reconstrution algorithms. It is a three dimensional model of a human head consisting of 10 ellipsoids (see Shepp [9]) which were shrinked here to fit into the unit sphere S^2. Figure 1 shows a cut at $x_3 = 0.2721$.

Let $a_1^{(j)}, a_2^{(j)}, a_3^{(j)}$, $j = 1, \ldots, 10$, denote the axes of the j-th ellipsoid, $d^{(j)}$ denote its density value and $s_2^{(j)} - s_1^{(j)}$ the diameter of the ellipsoid in the direction of $t \in S^2$. Since the Radon transform is linear, the Radon transform of the Shepp-Logan phantom can be calculated to be

$$\mathcal{R}F(s,t) = \sum_{j=1}^{10} \pi d^{(j)} a_1^{(j)} a_2^{(j)} a_3^{(j)} (s - s_1^{(j)})(s_2^{(j)} - s) \left(\frac{s_2^{(j)} - s_1^{(j)}}{2} \right)^{-3/2} \tag{5.1}$$

Figure 2 shows the reconstruction results according to formula (3.6) for Cesaro-means of index $k = 4$ and for Newman–Shapiro operators.

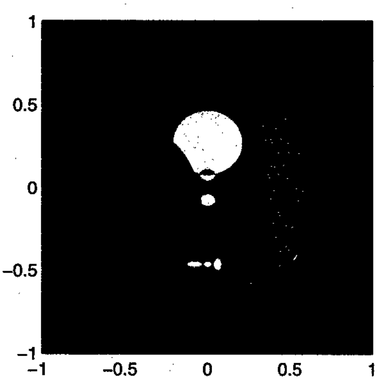

FIG. 1. Shepp-Logan phantom.

The values $k = 1.6$ and $k = 2$ were tested, too, but for high degrees of μ no convergent behaviour could be observed.

For Cesaro-means with $k = 4$ and for Newman-Shapiro operators Figure 2 clearly shows an improving behaviour of the reconstructions for increasing μ.

The Newman-Shapiro operators show a better convergence and for $\mu \geq 150$ even the small structures in the original head can be detected in the reconstruction. It can be expected that for higher degrees of μ this behaviour will become more evident.

Unfortunately, for $\mu \geq 170$ the computation of the coefficients $a_{\mu\nu}$ for the Newman-Shapiro operators caused some numerical problems so that the calculations were stopped with $\mu = 160$. Although the numerical results look quite promising, the drawback in the reconstruction is the computational time. For $\mu = 160$ the computation took 27.5 hours for the Radon transform and 31 hours for the evaluation at the points $x \in [-1, 1]^2$. The evaluation was done on an equidistant grid of 200×200 points.

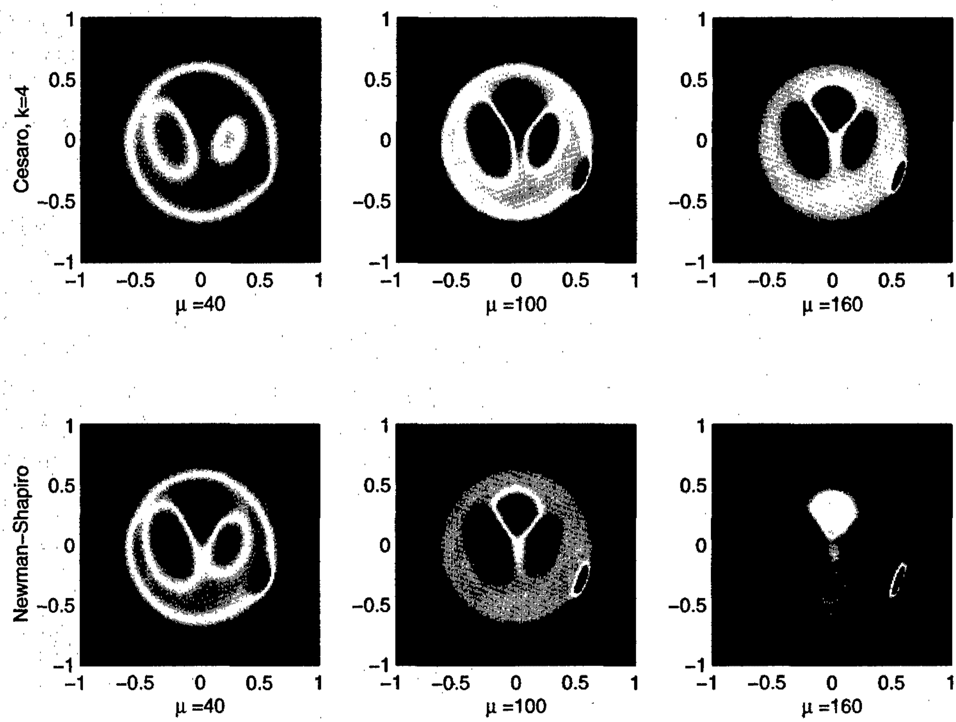

FIG. 2. reconstruction of the Shepp-Logan phantom.

In principle there is no problem to produce three dimensional reconstructions. The evaluation points x only have to be chosen from a grid in $[-1,1]^3$. Because of the time consuming calculations this was not done here, yet.

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